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LETTERS TO THE EDITOR

MASS AND STIFFNESS FIXED POINTS OF VIBRATIONAL DISCRETE SYSTEMS

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1. INTRODUCTION

The occurrence of fixed points in the frequency responses of vibrating mechanical systems may complicate the reduction of vibration amplitudes. Therefore, it is recommended that the system fixed points are determined before starting vibration control. There are three types of fixed points: mass, stiffness and damping fixed points. The intersection of all amplitude curves which occur in a frequency response of a vibrating system during a parametric variation of a mass, is defined as a mass fixed point. At the frequency where this point occurs, the vibration amplitudes are independent of the values of the mass. Damping and stiffness fixed points are similarly defined. Note, however, that the frequency responses of systems with finite or infinite degrees of freedom possess damping fixed points if only a single damper is connected [1]. Also, if the system includes two or more dampers then the response curves do not show damping fixed points, since the response of the system cannot be expressed in a bilinear form.

Damping fixed points of systems with one and two degrees of freedom were treated in connection with vibration absorption and vibration isolation by many authors, including Den Hartog [2] and Klotter [3]. Bogy and Paslay [4] used the damping fixed points to obtain optimal damping for the purpose of minimizing the maximum steady state response of a particular linear damped two-degree-of-freedom vibratory system. Henney and Raney [1] used the damping fixed points to find approximate analytical expressions for optimum damping for a uniform beam forced and damped in four different configurations.

Mass, stiffness, and damping fixed points of a system with two degrees of freedom were considered by Abu-Hilal [5], where the frequencies at which damping, mass, and/or stiffness fixed points occur and their amplitudes were determined analytically.

In this paper, the mass and stiffness fixed points of vibratory discrete linear systems with n degrees of freedom are investigated. A procedure for determining the frequencies at which mass and/or stiffness fixed points occur, is presented. To verify this procedure, all frequencies at which mass and/or stiffness fixed points of a vibrating three-mass system occur are determined in closed forms.

2. PROCEDURE FOR DETERMINING THE MASS AND STIFFNESS FIXED POINTS OF SYSTEMS WITH *N* DEGREES OF FREEDOM

An undamped system is assumed in this paper since mass and stiffness fixed points occur perfectly in a frequency response only in the absence of damping. The damping is usually

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represented by a damping ratio ζ_i, which is a function of the system parameter c_i, k_i, and m_i. If one or more of these parameters are changed, then the damping ratio is changed also. Consequently, by varying the masses and/or the stiffnesses of the system, each response curve will have a different damping ratio. As a result, the amplitudes of the different response curves will be differently damped and reduced at the same frequency, where a mass and/or a stiffness fixed point occurs. Therefore, in this case all the existing fixed points will be scattered. The equation of motion of an undamped system with n degrees of freedom is given as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{K}$, $\mathbf{x}$, and $\mathbf{F}$ are the mass matrix of the system, the stiffness matrix of the system, the displacement vector of the system, and the excitation force vector respectively. If a harmonic force vector $\mathbf{F} = \mathbf{P}\cos\omega t$ is assumed, where $\mathbf{P} = [P_1, \ldots, P_n]^{\mathrm{T}}$ is the vector of the force amplitudes and ω is the circular excitation frequency, then the displacement vector of the system is obtained by using the solution

$$\mathbf{x} = \mathbf{X}\cos\omega t. \tag{2}$$

Substituting equation (2) in equation (1) and simplifying yields

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{X} = \mathbf{P}, \tag{3}$$

where $\mathbf{X} = [X_1, \ldots, X_n]^{\mathrm{T}}$ is the vector of the displacement amplitudes, $X_j = X_{j1} + \cdots + X_{jn}$ is the vibration amplitude of mass j, and X_{jg} $(j, g = 1, \ldots, n)$ is the frequency response of mass j due to a force P_g applied at position g, with all other forces equal to zero (i.e., $P_1 = \cdots = P_{g-1} = P_{g+1} = \cdots = P_n = 0$).

The frequency response X_{jg} can be obtained from equation (3) and written as

$$X_{jg} = \frac{P_g}{\Delta}(a_1 e_i + a_2), \tag{4}$$

where Δ is the determinant of the matrix $\mathbf{K} - \omega^2\mathbf{M}$, e_i represents a stiffness k_i or a mass m_i, and a_1 and a_2 are polynomials in the variable ω^2. That is:

$$a_1 = \alpha_p\omega^{2p} + \cdots + \alpha_1\omega^2 + \alpha_0, \tag{5}$$

$$a_2 = \beta_q\omega^{2q} + \cdots + \beta_1\omega^2 + \beta_0, \tag{6}$$

where α_i $(i = 0, \ldots, p)$ and β_j $(j = 0, \ldots, q)$ are constants and p and q are positive integers. In addition, the determinant Δ can be written as

$$\Delta = a_3 e_i + a_4. \tag{7}$$

Similarly, a_3 and a_4 are polynomials in the variable ω^2. Note that a_1 or a_2 may be zero, but a_3 and a_4 do not always equal zero. Substituting equation (7) in equation (4) yields the frequency response X_{jg} in a bilinear form

$$X_{jg} = P_g\frac{(a_1 e_i + a_2)}{(a_3 e_i + a_4)}. \tag{8}$$

The response X_{jg} is independent of e_i if

$$\frac{a_1}{a_3} = \frac{a_2}{a_4}. \tag{9}$$

That is, by varying the values of e_i while all other parameters remain constant, all curves of the amplitude X_{jg} pass through fixed points, whatever the value of that e_i may be. These fixed points are determined by equating X_{jg} to two different values of e_i. For simplification, the values 0, 1, and ∞ will be used with the following three cases:

Case 1: $a_1 \neq 0$ and $a_2 \neq 0$

In this case, the values $e_i = 0$ and ∞ can always be used. Equating $X_{jg}(e_i = 0) = P_g a_2/a_4$ and $X_{jg}(e_i = \infty) = P_g a_1/a_3$ yields

$$a_1 a_4 - a_2 a_3 = 0. \tag{10}$$

Solving this equation, which is a polynomial in the variable ω^2, yields the frequencies at which fixed points occur.

Case 2: $a_1 = 0$ and $a_2 \neq 0$

Here, the value $e_i = 0$ can always be used and the value of $e_i = \infty$ can also be used. However, in general, the frequencies of the fixed points cannot all be obtained. Therefore, the value $e_i = 1$ instead of the value $e_i = \infty$ has to be used to achieve all the fixed points. Equating $X_{jg}(e_i = 0) = P_g a_2/a_4$ and $X_{jg}(e_i = 1) = P_g a_2/(a_3 + a_4)$ yields

$$a_2 a_3 = 0. \tag{11}$$

Solving this equation yields all frequencies at which fixed points occur.

Case 3: $a_1 \neq 0$ and $a_2 = 0$

The value $e_i = \infty$ can always be used in this case, and $e_i = 0$ can also be used. However, in general, the frequencies of the fixed points cannot all be obtained. Therefore, the value $e_i = 1$ instead of $e_i = 0$ has to be used to obtain all fixed points. Equating $X_{jg}(e_i = \infty) = P_g a_1/a_3$ and $X_{jg}(e_i = 1) = P_g a_1/(a_3 + a_4)$ yields

$$a_1 a_4 = 0. \tag{12}$$

Solving this equation yields all frequencies at which fixed points occur.

Equations (10)–(12) can also be used to determine the fixed points of damped systems. In this case, a_1–a_4 are functions of ω and become, in general, complex because of the damping. Equation (10) becomes, in this case,

$$a_1 a_4 \pm a_2 a_3 = 0 \tag{13}$$

while equations (11) and (12) remain unchanged.

Note that, equations (10)–(12) can always be solved analytically only if $n \leqslant 3$. However, with $n > 3$, these equations are, in general, solved numerically.

3. EXAMPLE: MASS AND STIFFNESS FIXED POINTS OF A THREE-MASS SYSTEM

To verify the above procedure, a system with three degrees of freedom as shown in Figure 1 is presented. The equations of motion of the system are

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_1, \tag{14}$$

$$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 - k_3 x_3 = F_2, \tag{15}$$

$$m_3 \ddot{x}_3 - k_3 x_2 + k_3 x_3 = F_3. \tag{16}$$

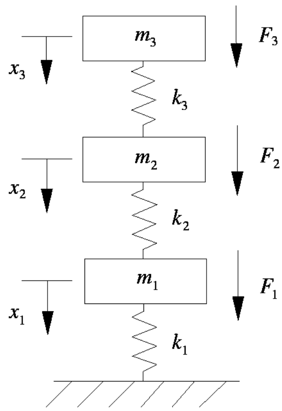

Figure 1. Three-degrees-of-freedom system.

If the acting forces are harmonic ($F_i = F_0 \cos \omega t$), then the frequency responses X_{jg} ($j, g = 1, \ldots, 3$) can be obtained by substituting the solution $x_i = X_i \cos \omega t$ in equations (14)–(16). Thus

$$X_{11} = c[m_2 m_3 \omega^4 - (k_2 m_3 + k_3 m_3 + k_3 m_2)\omega^2 + k_2 k_3], \tag{17}$$

$$X_{21} = c(-k_2 m_3 \omega^2 + k_2 k_3), \tag{18}$$

$$X_{31} = c k_2 k_3, \tag{19}$$

$$X_{12} = c(-k_2 m_3 \omega^2 + k_2 k_3), \tag{20}$$

$$X_{22} = c[m_1 m_3 \omega^4 - (k_1 m_3 + k_2 m_3 + k_3 m_1)\omega^2 + k_1 k_3 + k_2 k_3], \tag{21}$$

$$X_{32} = c(-k_3 m_1 \omega^2 + k_1 k_3 + k_2 k_3), \tag{22}$$

$$X_{13} = c k_2 k_3, \tag{23}$$

$$X_{23} = c(-k_3 m_1 \omega^2 + k_1 k_3 + k_2 k_3), \tag{24}$$

$$X_{33} = c[m_1 m_2 \omega^4 - [m_2(k_1 + k_2) + m_1(k_2 + k_3)]\omega^2 + k_1 k_2 + k_1 k_3 + k_2 k_3], \tag{25}$$

where

$$c = F_0/\Delta, \tag{26}$$

$$\Delta = -m_1 m_2 m_3 \omega^6 + [m_2 m_3 (k_1 + k_2) + m_1 m_3 (k_2 + k_3) + k_3 m_1 m_2]\omega^4$$
$$- [k_1 k_2 m_3 + k_3(k_1 m_3 + k_1 m_2 + k_2 m_3 + k_2 m_2 + k_2 m_1)]\omega^2 + k_1 k_2 k_3. \tag{27}$$

For example, in order to determine the fixed points of the frequency response X_{21} by varying the values of m_1, the amplitude X_{21} can be written, according to equation (8), as

$$X_{21} = F_0 \frac{a_1 m_1 + a_2}{a_3 m_1 + a_4},$$

TABLE 1

Frequencies of the mass and/or stiffness fixed points of the three-mass system shown in Figure 1

	Varying parameter					
	k_1	k_2	k_3	m_1	m_2	m_3
X_{11}	f_{10}, f_{11}	f_1, f_5	f_1	f_1, f_{10}, f_{11}	f_1, f_3	f_1
X_{21}	f_3, f_{10}, f_{11}	f_1, f_2, f_3, f_5	f_1, f_4	f_1, f_3, f_{10}, f_{11}	f_1, f_3, f_4	f_1, f_4
X_{31}	f_{10}, f_{11}	f_1, f_2, f_5	f_1, f_6, f_7	f_1, f_{10}, f_{11}	f_1, f_3, f_4	f_1, f_8, f_9
X_{12}	f_3, f_{10}, f_{11}	f_1, f_2, f_3, f_5	f_1, f_4	f_1, f_3, f_{10}, f_{11}	f_1, f_3, f_4	f_1, f_4
X_{22}	f_3	f_2, f_3	f_1, f_4	f_1, f_3	f_1, f_3, f_4	f_1, f_4
X_{32}	f_3	f_2, f_3	f_1, f_4, f_6, f_7	f_1, f_3	f_1, f_3, f_4	f_1, f_4, f_8, f_9
X_{13}	f_{10}, f_{11}	f_1, f_2, f_5	f_1, f_6, f_7	f_1, f_{10}, f_{11}	f_1, f_3, f_4	f_1, f_8, f_9
X_{23}	f_3	f_2, f_3	f_1, f_4, f_6, f_7	f_1, f_3	f_1, f_3, f_4	f_1, f_4, f_8, f_9
X_{33}	—	f_2	f_6, f_7	f_1	f_1, f_4	f_1, f_8, f_9

where in this case (case 2):

$$a_1 = 0,$$

$$a_2 = k_2(k_3 - m_3\omega^2),$$

$$a_3 = -m_2 m_3 \omega^6 + (k_2 m_3 + k_3 m_3 + k_3 m_2)\omega^4 - k_2 k_3 \omega^2, \text{ and}$$

$$a_4 = (k_1 + k_2) m_2 m_3 \omega^4 - [k_1 k_2 m_3 + (m_2 + m_3)(k_1 k_3 + k_2 k_3)]\omega^2 + k_1 k_2 k_3.$$

Equating X_{21} to the values $m_1 = 0$ and 1 yields

$$a_2 a_3 = k_2(k_3 - m_3\omega^2)(-\omega^2)[m_2 m_3 \omega^4 - (k_2 m_3 + k_3 m_3 + k_3 m_2)\omega^2 + k_2 k_3] = 0.$$

Solving this equation yields

$$\omega_1^2 = 0,$$

$$\omega_2^2 = k_3/m_3, \quad \text{and}$$

$$\omega_{3,4}^2 = \frac{1}{2m_2 m_3}\left[k_2 m_3 + k_3 m_3 + k_3 m_2 \pm \sqrt{(k_2 m_3 + k_3 m_3 + k_3 m_2)^2 - 4k_2 k_3 m_2 m_3}\right],$$

where

$$\omega_i = 2\pi f_i. \tag{28}$$

On the other hand, equating X_{21} to the values $m_1 = 0$ and ∞ yields

$$a_2 = k_2(k_3 - m_3\omega^2) = 0.$$

Solving this equation gives, however, only the second frequency ω^2.

Table 1 contains all frequencies at which stiffness and/or mass fixed points in the frequency responses X_{jg} occurred. These frequencies can be obtained similar to those of X_{21}.

The frequencies f_i in Table 1 are defined as follows:

$$f_1 = 0, \tag{29}$$

$$f_2 = \frac{1}{2\pi}\sqrt{k_1/m_1}, \tag{30}$$

$$f_3 = \frac{1}{2\pi}\sqrt{k_3/m_3}, \tag{31}$$

$$f_4 = \frac{1}{2\pi}\sqrt{(k_1 + k_2)/m_1}, \tag{32}$$

$$f_5 = \frac{1}{2\pi}\sqrt{k_3(m_2 + m_3)/(m_2 m_3)}, \tag{33}$$

$$f_{6,7} = \frac{1}{2\pi}\sqrt{\frac{1}{2m_1 m_2}[b_1 \pm \sqrt{b_1^2 - b_2}]}, \tag{34}$$

$$f_{8,9} = \frac{1}{2\pi}\sqrt{\frac{1}{2m_1 m_2}[b_3 \pm \sqrt{b_3^2 - b_4}]}, \tag{35}$$

$$f_{10,11} = \frac{1}{2\pi}\sqrt{\frac{1}{2m_2 m_3}[b_5 \pm \sqrt{b_5^2 - b_6}]}, \tag{36}$$

where

$$b_1 = k_1 m_2 + k_2 m_1 + k_2 m_2, \tag{37}$$

$$b_2 = 4k_1 k_2 m_1 m_2, \tag{38}$$

$$b_3 = k_1 m_2 + k_2 m_1 + k_2 m_2 + k_3 m_1, \tag{39}$$

$$b_4 = 4(k_1 k_2 + k_1 k_3 + k_2 k_3) m_1 m_2, \tag{40}$$

$$b_5 = k_2 m_3 + k_3 m_2 + k_3 m_3, \tag{41}$$

$$b_6 = 4k_2 k_3 m_2 m_3. \tag{42}$$

As shown in Table 1, the frequency responses X_{jg} possess the same fixed points as X_{gj}, because of the symmetry of the matrix $\mathbf{K} - \omega^2\mathbf{M}$. Furthermore, from Table 1, the following points can be observed:

(1) at the frequencies f_2, f_5, f_6, and f_7 only stiffness fixed points occur,
(2) at the frequencies f_8 and f_9 only mass fixed points occur, and
(3) at the frequencies f_1, f_3, f_4, f_{10}, and f_{11} mass and stiffness fixed points occur.

If the frequency response X_{21} is to be considered, it can be observed that at the frequency f_3 stiffness field points occur corresponding to k_1 and k_2 and mass fixed points occur corresponding to m_1 and m_2. This means that a reduction of the amplitude X_{21} at the frequency f_3 cannot be controlled by varying the values of k_1, k_2, m_1, and/or m_2.

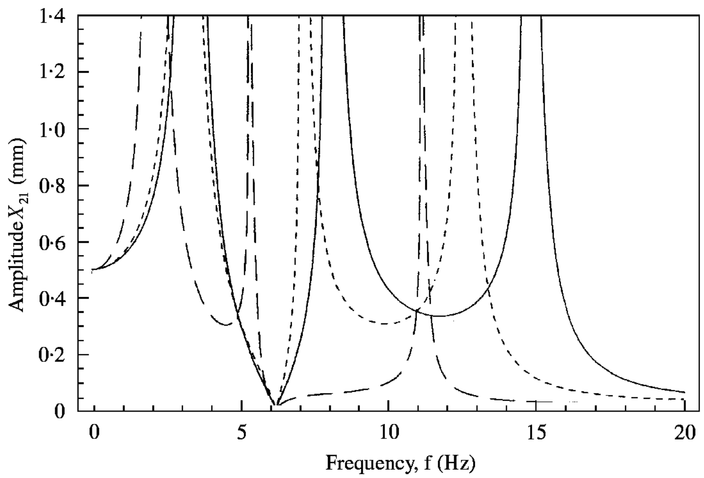

Figure 2. Frequency response X_{21} for different values of m_1. —— $m_1 = 80$ kg, ····· $m_1 = 160$ kg, - - - - - $m_1 = 800$ kg.

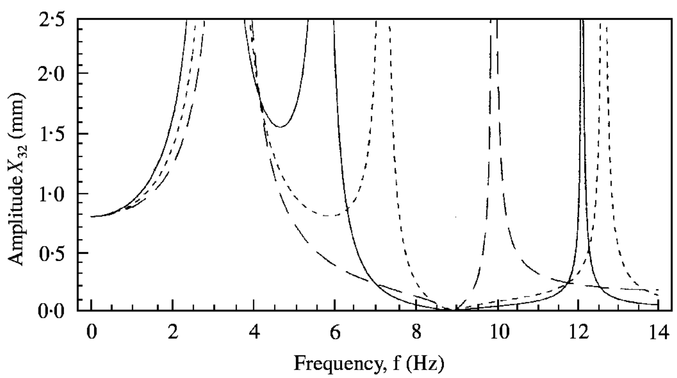

Figure 3. Frequency response X_{32} for different values of k_3. —— $k_3 = 60$ kN/m, ····· $k_3 = 120$ kN/m, - - - - - $k_3 = 600$ kN/m.

In order to make the results obtained for the three-mass system illustrative, some frequency responses are given graphically, based on the data: $m_1 = 160$ kg, $m_2 = 100$ kg, $m_3 = 80$ kg, $k_1 = 200$ kN/m, $k_2 = 300$ kN/m, $k_3 = 120$ kN/m, and $F_0 = 100$ N.

Figure 2 shows the frequency response X_{21} for different values of m_1 of 80, 160, and 800 kg. The frequencies at which mass fixed points in the frequency response X_{21} occur, can be obtained from Table 1 and equations (29), (31), and (36). These frequencies are: $f_1 = 0$ Hz, $f_3 = 6{\cdot}164$ Hz, $f_4 = 10{\cdot}97$ Hz, and $f_{10} = 4{\cdot}897$ Hz. Comparison of these values with those in Figure 2, shows that they coincide with each other.

Figure 3 shows the frequency response X_{32} for different values of the stiffness k_3 of 60, 120, and 600 kN/m. The frequencies at which stiffness fixed points in X_{32} occur can be obtained from the Figure; these frequencies are: $f_1 = 0$ Hz, $f_4 = 8{\cdot}897$ Hz, $f_6 = 4{\cdot}18$ Hz, and $f_7 = 11{\cdot}73$ Hz.

Figure 4 shows the frequency response X_{32} for different values of the stiffness k_2 of 150, 300, and 1500 kN/m. The frequencies at which fixed points in X_{32} occur, can be obtained from the Figure; these are: $f_2 = 5{\cdot}63$ Hz and $f_3 = 6{\cdot}164$ Hz.

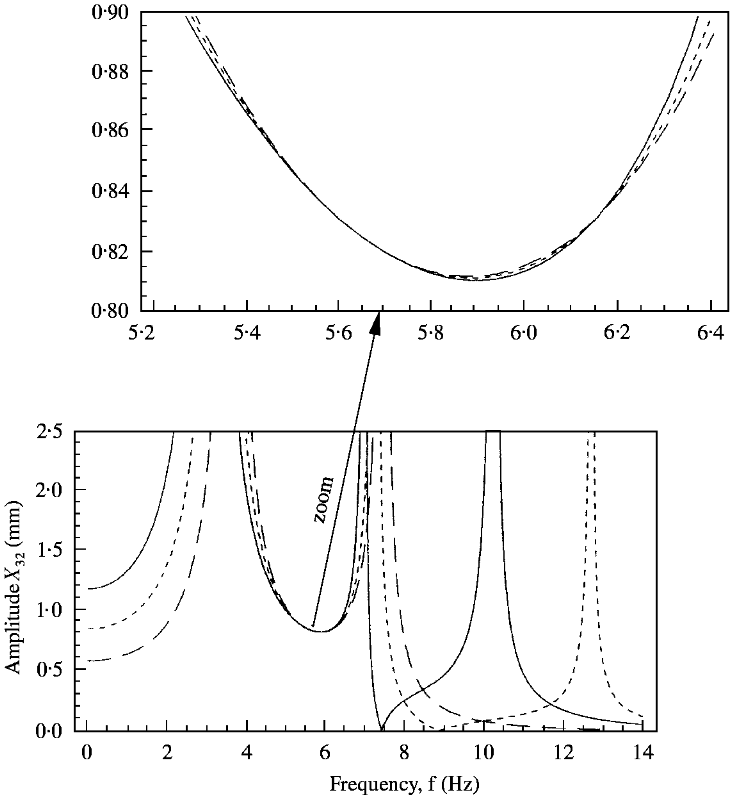

Figure 4. Frequency response X_{32} for different values of k_2. —— $k_2 = 150$ kN/m, ····· $k_2 = 300$ kN/m, - - - - - $k_2 = 1500$ kN/m.

4. CONCLUSIONS

The occurrence of fixed points in a frequency response of a vibrating system may complicate the vibration reduction, since the amplitudes of vibration near mass (stiffness) fixed points cannot be effectively reduced by varying the values of masses (stiffnesses). Therefore, it is recommended that these fixed points are determined before beginning vibration reduction. To obtain these fixed points, the procedure presented in this paper can be used.

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